



Section (4.5) :

4.5 THE DEFINITE INTEGRAL

RIEMANN SUMS AND THE DEFINITE INTEGRAL

4.5.1 DEFINITION A function f is said to be *integrable* on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x) dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

which is called the *definite integral* of f from a to b . The numbers a and b are called the *lower limit of integration* and the *upper limit of integration*, respectively, and $f(x)$ is called the *integrand*.

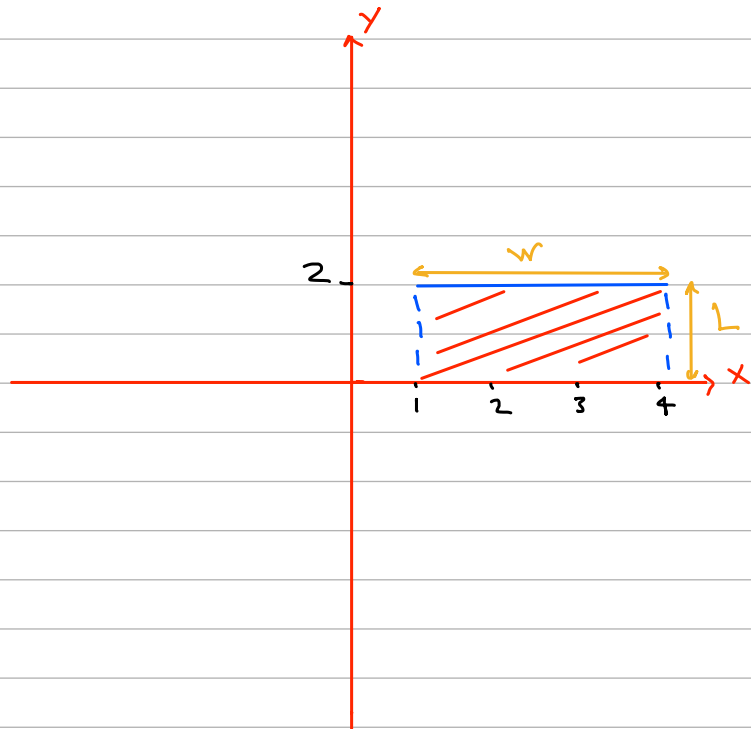
4.5.2 THEOREM If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x) dx \quad (1)$$

► **Example 1** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2 \, dx$

$y = 2$



$\int_1^4 2 \, dx = \text{area of rectangular}$

$= \text{Length} \cdot \text{width}$

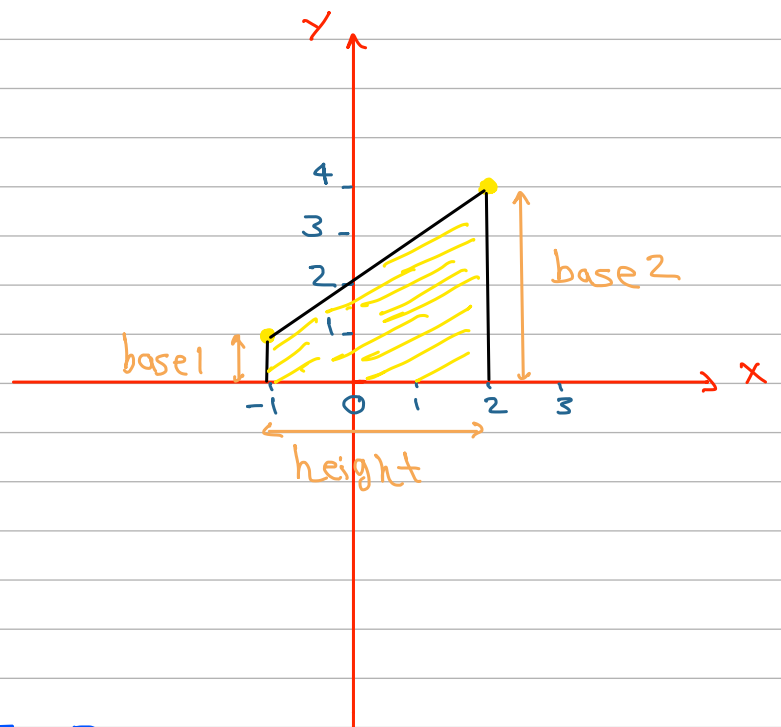
$= 2 \cdot 3 = 6$

$$(b) \int_{-1}^2 (x+2) dx$$

$$Y = X + 2$$

$$\text{at } X = -1 \rightarrow Y = -1 + 2 = 1 \quad (-1, 1)$$

$$\text{at } X = 2 \rightarrow Y = 2 + 2 = 4 \quad (2, 4)$$



$$\int_{-1}^2 (x+2) dx = \text{area of Trapezoid}$$

$$= \frac{1}{2} (\text{Base1} + \text{Base2}) \cdot \text{height}$$

$$= \frac{1}{2} (1 + 4) \cdot 3$$

$$= \frac{1}{2} (5)(3) = \frac{15}{2}$$

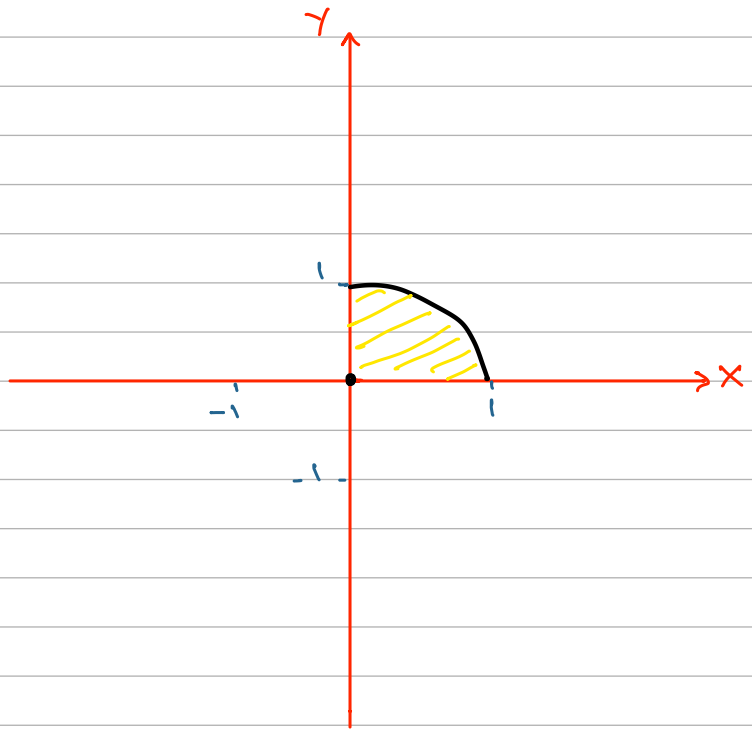
$$(c) \int_0^1 \sqrt{1-x^2} dx$$

$$y = \sqrt{1-x^2}$$

$$(y)^2 = (\sqrt{1-x^2})^2$$

$$y^2 = 1-x^2$$

$$y^2 + x^2 = 1$$



$$\int_0^1 \sqrt{1-x^2} dx = \text{area of quarter circle}$$

$$= \frac{1}{4} \pi r^2$$

$$= \frac{1}{4} \pi (1)^2$$

$$= \frac{\pi}{4}$$

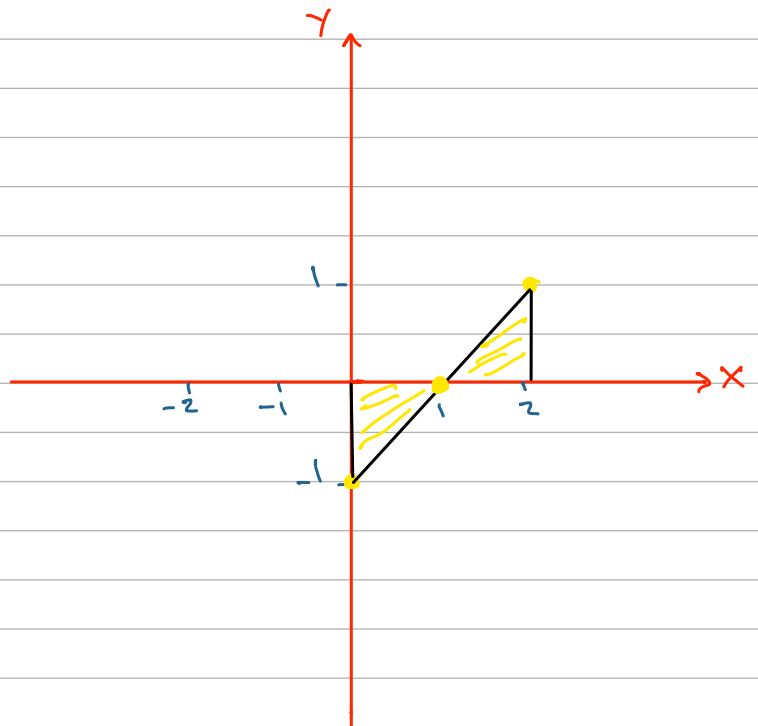
► **Example 2** Evaluate

(a) $\int_0^2 (x-1) dx$

$$Y = X - 1$$

at $X=0 \rightarrow Y=0-1=-1 \quad (0,-1)$

at $X=2 \rightarrow Y=2-1=1 \quad (2,1)$



$$\int_0^2 (x-1) dx = A_1 + (-A_2)$$

$$= A_1 - A_2$$

$$= \frac{1}{2} (\text{base1} \cdot \text{height1}) - \frac{1}{2} (\text{base2} \cdot \text{height2})$$

$$= \frac{1}{2} (1 \cdot 1) - \frac{1}{2} (1 \cdot 1)$$

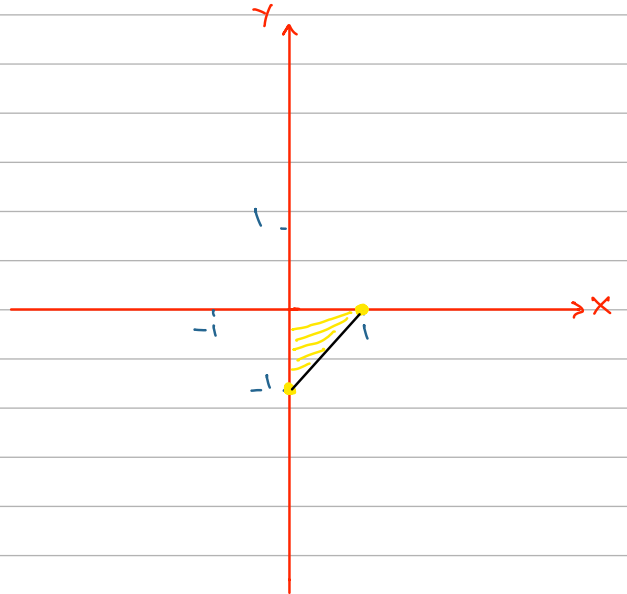
$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$(b) \int_0^1 (x-1) dx$$

$$Y = X - 1$$

$$\text{at } X=0 \rightarrow Y=0-1=-1 \quad (0,-1)$$

$$\text{at } X=1 \rightarrow Y=1-1=0 \quad (1,0)$$



$$\int_0^1 (x-1) dx = \frac{1}{2} (\text{base} \cdot \text{height})$$

$$= \frac{1}{2} (1 \cdot 1)$$

$$= \frac{1}{2}$$

■ PROPERTIES OF THE DEFINITE INTEGRAL

4.5.3 DEFINITION

(a) If a is in the domain of f , we define

$$\int_a^a f(x) dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

► Example 3

$$(a) \int_1^1 x^2 dx = \left(\frac{x^3}{3} \right)_1^1$$

$$= \left(\frac{1^3}{3} - \frac{1^3}{3} \right)$$

$$= \frac{1}{3} - \frac{1}{3} = 0$$

دائماً إذا كانت حدود التكامل متساوية إذا الناتج يساوي صفر

$$(b) \int_1^0 \sqrt{1-x^2} dx = - \int_0^1 \sqrt{1-x^2}$$

$$= - \frac{\pi}{4}$$

إذا كانت حدود التكامل الحد الذي فوق اصغره الحد الذي تحت إذا قلب حدود التكامل ونضع إشارة سالبة قبل التكامل

4.5.4 THEOREM If f and g are integrable on $[a, b]$ and if c is a constant, then cf , $f + g$, and $f - g$ are integrable on $[a, b]$ and

$$(a) \int_a^b cf(x) dx = c \int_a^b f(x) dx$$

$$(b) \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$(c) \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

► **Example 4** Evaluate

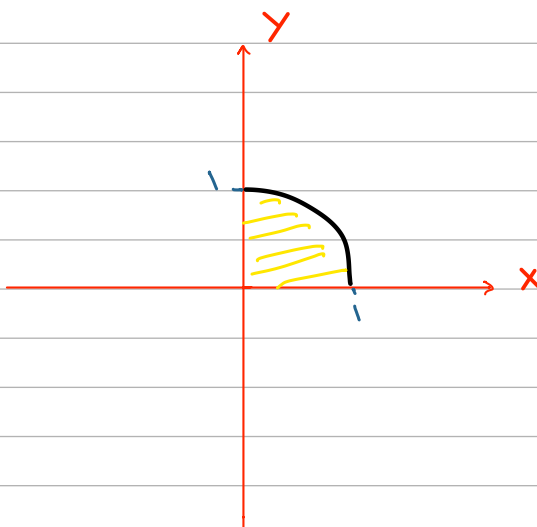
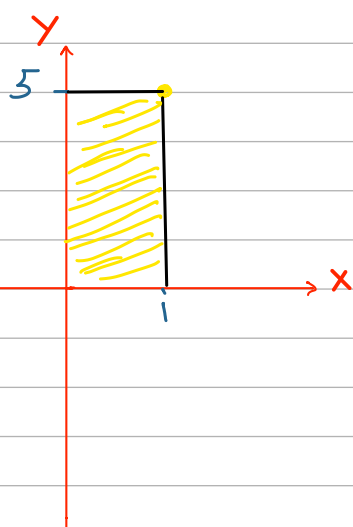
$$\int_0^1 (5 - 3\sqrt{1-x^2}) dx$$

$$\int_0^1 5 dx - 3 \int_0^1 \sqrt{1-x^2} dx$$

$$\int_0^1 5 dx = (5x)'_0 = (5(1) - 5(0)) = 5$$

$$\int_0^1 \sqrt{1-x^2} dx = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}$$

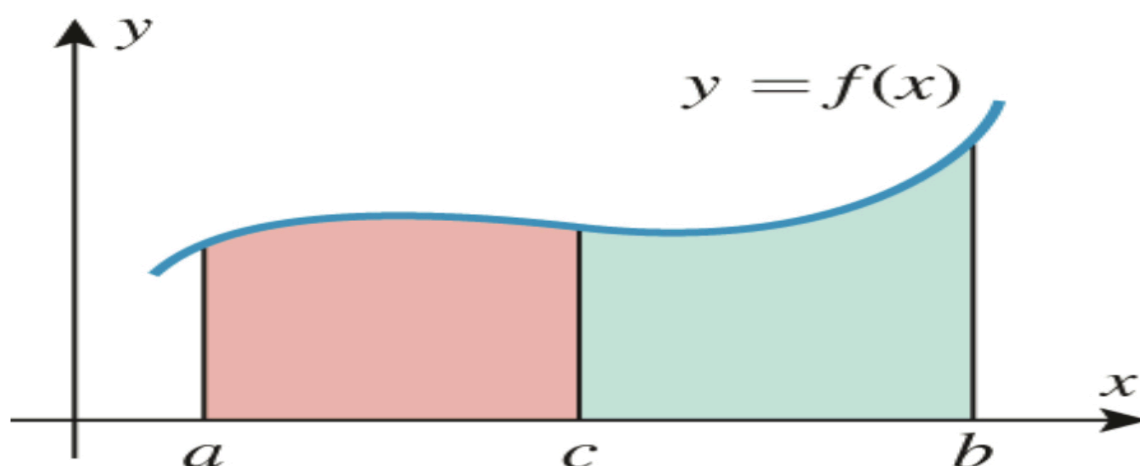
$$\int_0^1 5 dx - 3 \int_0^1 \sqrt{1-x^2} dx = 5 - 3\left(\frac{\pi}{4}\right)$$



4.5.5 THEOREM If f is integrable on a closed interval containing the three points a , b , and c , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

no matter how the points are ordered.



▲ **Figure 4.5.7**

4.5.6 THEOREM

(a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x) dx \geq 0$$

(b) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx$$