IQ ACADEMY

Section (4.9) :

4.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Recall from Section 4.3 that indefinite integrals of the form

$$\int f(g(x))g'(x)\,dx$$

can sometimes be evaluated by making the u-substitution

$$u = g(x), \quad du = g'(x)\,dx \tag{1}$$

which converts the integral to the form

$$\int f(u)\,du$$

To apply this method to a definite integral of the form

$$\int_a^b f(g(x))g'(x)\,dx$$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1.

First evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\,dx = \left[\int f(g(x))g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

Method 2.

Make the substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of u.

► Example 1 Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution :

method 1 :

$$\int_0^2 x(x^2+1)^3\,dx$$

$$u = x^2+1$$

$$du = 2x\,dx \longrightarrow \frac{du}{2} = x\,dx$$

$$\frac{1}{2}\int_0^2 u^3\,du$$

$$= \frac{1}{2}\left(\frac{u^4}{4}\right)_0^2$$

$$= \frac{1}{8}\left(u^4\right)_0^2$$

$$= \frac{1}{8}\left((x^2+1)^4\right)_0^2$$

$$= \frac{1}{8}\left((2^2+1)^4 - (0^2+1)^4\right)$$

$$= \frac{1}{8}\left(5^4 - 1^4\right)$$

$$= \frac{1}{8}(625-1)$$

$$= \frac{1}{8}(624) = 78$$

method :2 :

$$\int_0^2 x(x^2+1)^3\,dx$$

$$u = x^2+1$$

$$du = 2x\,dx \longrightarrow \frac{du}{2} = x\,dx$$

$$\frac{1}{2}\int_0^2 u^3\,du$$

$$\text{at } x = 0 \longrightarrow u = 0^2+1 = 1$$

$$\text{at } x = 2 \longrightarrow u = 2^2+1 = 5$$

$$\frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\left(\frac{u^4}{4}\right)_1^5$$

$$= \frac{1}{8}\left(u^4\right)_1^5$$

$$= \frac{1}{8}\left(5^4 - 1^4\right)$$

$$= \frac{1}{8}(625 - 1)$$

$$= \frac{1}{8}(624) = 78$$

4.9.1 THEOREM *If g' is continuous on $[a,b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

▶ **Example 2** Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x\,dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9\,dx$$

a) $u = \sin 2x$

$du = 2\cos 2x\,dx \longrightarrow \frac{du}{2} = \frac{\cancel{2}\cos 2x}{\cancel{2}}\,dx$

$\frac{du}{2} = \cos 2x\,dx$

at $x = 0 \longrightarrow u = \sin 0 = 0$

at $x = \pi/8 \longrightarrow u = \sin\left(\frac{2\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right) = \sqrt{2}/2$

$\int_0^{\sqrt{2}/2} u^5 \frac{du}{2} = \frac{1}{2}\int_0^{\sqrt{2}/2} u^5\,du$

$= \frac{1}{2}\left(\frac{u^6}{6}\right)_0^{\sqrt{2}/2} = \frac{1}{12}\left(u^6\right)_0^{\sqrt{2}/2}$

$= \frac{1}{12}\left(\left(\frac{\sqrt{2}}{2}\right)^6 - (0)^6\right)$

$= \frac{1}{12}\cdot\frac{(2^{1/2})^6}{64} = \frac{1}{12}\cdot\frac{2^3}{64} = \frac{1}{12}\cdot\frac{8}{64}$

$= \frac{1}{96}$

b) $\int_2^5 (2x-5)(x-3)^9 \, dx$

$u = x-3$

$du = dx$

$u = x-3 \longrightarrow x = u+3$

at $x=2 \longrightarrow u = 2-3 = -1$

at $x=5 \longrightarrow u = 5-3 = 2$

$\int_{-1}^{2} (2(u+3)-5)(u)^9 \, du$

$\int_{-1}^{2} (2u+6-5)(u)^9 \, du$

$\int_{-1}^{2} (2u+1)(u)^9 \, du$

$\int_{-1}^{2} (2u^{10}+u^9) \, du$

$= \left(2\frac{u^{11}}{11} + \frac{u^{10}}{10}\right)_{-1}^{2}$

$= \left(2\frac{2^{11}}{11} + \frac{2^{10}}{10}\right) - \left(2\frac{-1^{11}}{11} + \frac{-1^{10}}{10}\right)$

$= \left(\frac{4096}{11} + \frac{1024}{10}\right) - \left(\frac{-2}{11} + \frac{1}{10}\right)$

$= \frac{40960+11264}{110} - \frac{-20+11}{110} = \frac{52233}{110}$

$= \frac{52233}{110}$

► **Example 3** Evaluate $\int_1^3 \frac{\cos(\pi/x)}{x^2}\,dx$.

$u = \pi/x$

$du = -\frac{\pi}{x^2}dx \longrightarrow \frac{du}{-\pi} = \frac{1}{x^2}dx$

at $x = 1 \longrightarrow u = \pi/1 = \pi$

at $x = 3 \longrightarrow u = \pi/3$

$$\int_{\pi}^{\pi/3} \cos(u) \cdot -\frac{du}{\pi}$$

$$= -\frac{1}{\pi}\int_{\pi}^{\pi/3} \cos(u)\,du$$

$$= -\frac{1}{\pi}\Big(\sin u\Big)_{\pi}^{\pi/3}$$

$$= -\frac{1}{\pi}\Big(\sin(\pi) - \sin(\pi/3)\Big)$$

$$= -\frac{1}{\pi}\left(\frac{\sqrt{3}}{2} - 0\right)$$

$$= -\frac{1}{\pi} \cdot \frac{\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{2\pi}$$