Section (4.6) :

THE FUNDAMENTAL THEOREM OF CALCULUS

4.6.1 THEOREM ***(The Fundamental Theorem of Calculus, Part 1)*** *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

► **Example 1** Evaluate $\int_1^2 x\,dx$.

$$\int_1^2 x\,dx = \left(\frac{x^2}{2}\right)\Big|_1^2 = \left(\frac{2^2}{2}\right) - \left(\frac{1^2}{2}\right)$$

$$= \frac{4}{2} - \frac{1}{2}$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$

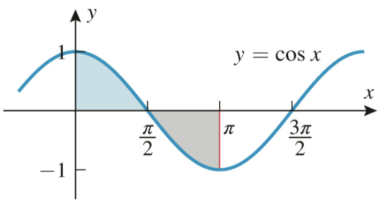
► **Example 2** $\int_0^3 (9 - x^2)\,dx$

$$= \left(9x - \frac{x^3}{3}\right)\Big|_0^3$$

$$= \left(9(3) - \frac{3^3}{3}\right) - \left(9(0) - \frac{0^3}{3}\right)$$

$$= (27 - 9) - (0)$$

$$= 18$$



▲ **Figure 4.6.4**

▶ Example 3

(a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 4.6.4).
(b) Make a conjecture about the value of the integral

$$\int_0^{\pi} \cos x \, dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

a) $A = \int_0^{\pi/2} \cos x \, dx$

$= (\sin x)_0^{\pi/2}$

$= (\sin(\pi/2) - \sin(0))$

$= 1 - 0 = 1$

b) $\int_0^{\pi} \cos x \, dx = A_1 + (-A_2)$

$= A_1 - A_2 = 0$

$\int_0^{\pi} \cos x \, dx = (\sin x)_0^{\pi}$

$= (\sin(\pi) - \sin(0))$

$= 0$

THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Let F be any antiderivative of f on $[a, b]$ and let C denote an arbitrary constant. Then

$$\int f(x)\,dx\Big]_a^b = (F(x)+C)\Big]_a^b = (F(b)+C) - (F(a)+C) = F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\int_a^b f(x)dx = [F(x)+C]_a^b$$

$$= [F(b)+C] - [F(a)+C]$$

$$= F(b) \cancel{+C} - F(a) \cancel{-C}$$

$$= F(b) - F(a)$$

▶ **Example 4** Table 4.2.1 will be helpful for the following computations.

$$\int_1^9 \sqrt{x}\,dx = \frac{x^{3/2}}{3/2}\Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9$$

$$= \frac{2}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{3}(27 - 1)$$

$$= \frac{52}{3}$$

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^2 (x^{1/2})\,dx$$

$$= \int_4^9 x^{5/2}\,dx$$

$$= \frac{x^{7/2}}{7/2}\Big]_4^9 = \frac{2}{7}x^{7/2}\Big]_4^9$$

$$= \frac{2}{7}\left(9^{7/2} - 4^{7/2}\right)$$

$$= \frac{2}{7}(2187 - 128)$$

$$= \frac{4118}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = \frac{1}{5}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{1}{5}\cdot - \cos x\Big]_0^{\pi/2}$$

$$= -\frac{1}{5}\left(\cos(\pi/2) - \cos(0)\right)$$

$$= -\frac{1}{5}(0 - 1)$$

$$= \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x \Big]_0^{\pi/3}$$

$$= \tan(\pi/3) - \tan(0)$$

$$= \sqrt{3} - 0$$

$$= \sqrt{3}$$

$$\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx = \sec x \Big]_{-\pi/4}^{\pi/4}$$

$$= \sec(\pi/4) - \sec(-\pi/4)$$

$$= \sqrt{2} - \sqrt{2}$$

$$= 0$$

▸ Example 5

$$\int_1^1 x^2\,dx = 0$$

$$\int_4^0 x\,dx = -\int_0^4 x\,dx$$

$$= -\frac{x^2}{2}\Big]_0^4$$

$$= -\left(\frac{4^2}{2} - \frac{0^2}{2}\right)$$

$$= -\left(\frac{16}{2} - 0\right)$$

$$= -8$$

► **Example 6** Evaluate $\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

$$\int_0^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

$$= \frac{x^3}{3}\Big]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3$$

$$= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) + \left(\left(\frac{3(3)^2}{2} - 2(3)\right) - (6-4)\right)$$

$$= \left(\frac{8}{3} - 0\right) + \left(\left(\frac{27}{2} - 6\right) - 2\right)$$

$$= \frac{8}{3} + \left(\frac{27-12}{2} - 2\right)$$

$$= \frac{8}{3} + \left(\frac{15}{2} - 2\right)$$

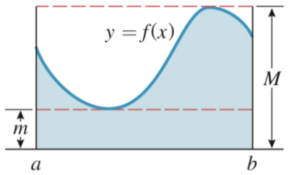
$$= \frac{8}{3} + \left(\frac{15-4}{2}\right)$$

$$= \frac{8}{3} + \frac{11}{2}$$

$$= \frac{16+33}{6}$$

$$= \frac{49}{6}$$

THE MEAN-VALUE THEOREM FOR INTEGRALS



▲ **Figure 4.6.9**

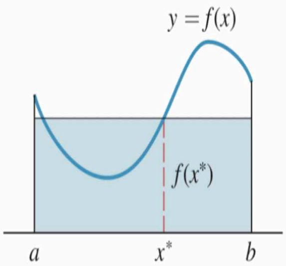
Let f be a continuous nonnegative function on $[a, b]$, and let m and M be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights m and M over the interval $[a, b]$ (Figure 4.6.9). It is clear geometrically from this figure that the area

$$A = \int_a^b f(x)\,dx$$

under $y = f(x)$ is at least as large as the area of the rectangle of height m and no larger than the area of the rectangle of height M. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f(x^*)$ between m and M whose area is precisely A; that is,

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

(Figure 4.6.10). This is a special case of the following result.



The area of the shaded rectangle is equal to the area of the shaded region in Figure 4.6.9.

▲ **Figure 4.6.10**

4.6.2 THEOREM (*The Mean-Value Theorem for Integrals*) *If f is continuous on a closed interval $[a, b]$, then there is at least one point x^* in $[a, b]$ such that*

$$\int_a^b f(x)\,dx = f(x^*)(b-a) \tag{8}$$

▶ **Example 8** Since $f(x) = x^2$ is continuous on the interval $[1, 4]$, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in $[1, 4]$ such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

$$\int_1^4 x^2\,dx = (x^*)^2(4-1)$$

$$\left(\frac{x^3}{3}\right)_1^4 = (x^*)^2(3)$$

$$\left(\frac{4^3}{3} - \frac{1^3}{3}\right) = 3x^{*2}$$

$$\left(\frac{64-1}{3}\right) = 3x^{*2}$$

$$\frac{63}{3} = 3x^{*2}$$

$$3x^{*2} = 21$$

$$x^{*2} = \frac{21}{3}$$

$$x^{*2} = 7 \longrightarrow \sqrt{x^{*2}} = \sqrt{7}$$

$$x^* = \pm\sqrt{7}$$

$$x^* = \pm 2.65$$

$$x^* = 2.65 \in [1, 4]$$

This is a special case of the following more general result, which applies even if f has negative values.

4.6.3 THEOREM (***The Fundamental Theorem of Calculus, Part 2***) *If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \tag{11}$$

Example 9 Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

by applying Part 2 of the Fundamental Theorem of Calculus, and then confirm the result by performing the integration and then differentiating.

$$\frac{d}{dx}\int_1^x t^3\,dt$$

$$= \frac{d}{dx}\left(\frac{t^4}{4}\right)_1^x$$

$$= \frac{d}{dx}\left(\frac{x^4}{4} - \frac{1}{4}\right)$$

$$= \frac{\cancel{4}x^3}{\cancel{4}} - 0 = x^3$$

$$\cancel{\frac{d}{dx}\int_1^x} t^3\,dt = x^3$$

▶ Example 10 Since
$$f(x) = \frac{\sin x}{x}$$
is continuous on any interval that does not contain the origin, it follows from (11) that on the interval $(0, +\infty)$ we have

$$\cancel{\frac{d}{dx}}\cancel{\int_1^x} \frac{\sin t}{t}\, \cancel{dt} = \frac{\sin x}{x}$$

DIFFERENTIATION AND INTEGRATION ARE INVERSE PROCESSES

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus (4.6.1) implies that
$$\int_a^x f'(t)\, dt = f(x) - f(a)$$
which tells us that if the value of $f(a)$ is known, then the function f can be recovered from its derivative f' by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus (4.6.3) states that
$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$$