Section (4.3) :

4.3 INTEGRATION BY SUBSTITUTION

التكامل بالتعويض

u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \tag{1}$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{2}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (2) can be expressed as

$$\boxed{\int f(u)\,du = F(u) + C} \tag{3}$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

▶ **Example 1** Evaluate $\int (x^2 + 1)^{50} \cdot 2x\,dx$.

$$\int (x^2+1)^{50}\, 2x\, dx$$

$$u = x^2 + 1$$

$$du = 2x\, dx$$

$$\int u^{50}\, du$$

$$= \frac{u^{50+1}}{50+1} + C$$

$$= \frac{u^{51}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C$$

Guidelines for u-Substitution

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)\,dx$$

produces an integral that is expressed entirely in terms of u and its differential du. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace u by $g(x)$ to express your final answer in terms of x.

EASY TO RECOGNIZE SUBSTITUTIONS

The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.

▶ Example 2

$$\int \sin(x+9)\,dx =$$

$u = x + 9$

$du = dx$

$\int \sin(u)\,du$

$= -\cos(u) + C$

$= -\cos(x+9) + C$

$$\int (x-8)^{23}\,dx =$$

$u = x - 8$

$du = dx$

$\int u^{23}\,du$

$= \dfrac{u^{24}}{24} + C$

$= \dfrac{(x-8)^{24}}{24} + C$

▸ Example 3 Evaluate $\int \cos 5x\, dx$.

$u = 5x$

$du = 5\,dx \longrightarrow \dfrac{du}{5} = dx$

$\int \cos(u) \dfrac{du}{5}$

$= \dfrac{1}{5} \int \cos(u)\, du$

$= \dfrac{1}{5} \sin(u) + C$

$= \dfrac{1}{5} \sin(5x) + C$

▶ **Example 4**

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} =$$

$u = \frac{1}{3}x - 8$

$du = \frac{1}{3}dx \longrightarrow 3du = dx$

$\int \frac{3}{u^5}\,du$

$3\int u^{-5}\,du$

$= 3\frac{u^{-5+1}}{-5+1} + C$

$= 3\frac{u^{-4}}{-4} + C$

$= -\frac{3}{4}u^{-4} + C$

$= -\frac{3}{4}\cdot\frac{1}{u^4} + C$

$= -\frac{3}{4}\cdot\frac{1}{\left(\frac{1}{3}x - 8\right)^4} + C$

► **Example 5**

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right) dx =$$

$$\int \frac{1}{X} dX + \int \sec^2(\pi X)\, dX$$

$$= \ln|X| + \int \sec^2(\pi X)\, dX$$

$$\int \sec^2(\pi X)\, dX$$

$u = \pi X \quad dX$

$du = \pi\, dX \longrightarrow \frac{du}{\pi} = dX$

$$\int \sec^2(u) \frac{du}{\pi}$$

$$= \frac{1}{\pi} \int \sec^2(u)\, du$$

$$= \frac{1}{\pi} \tan(u) + C$$

$$= \frac{1}{\pi} \tan(\pi X) + C$$

$$\int \frac{1}{X} dX + \int \sec^2(\pi X)\, dX$$

$$= \ln|X| + \frac{1}{\pi} \tan(\pi X) + C$$

▸ Example 6 Evaluate $\int \sin^2 x \cos x \, dx$.

$u = \sin x$

$du = \cos x \, dx$

$\int u^2 \, du$

$= \frac{u^3}{3} + C$

$= \frac{\sin^3 x}{3} + C$

▸ Example 7 Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$.

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} \, dx$

$2\sqrt{x} \, du = dx$

$\int \frac{\cos u}{\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}} \, du$

$2\int \cos u \, du$

$= 2 \sin u + C$

$= 2 \sin \sqrt{x} + C$

► **Example 8** Evaluate $\int t^4 \sqrt[3]{3-5t^5}\, dt$.

$u = 3 - 5t^5$

$du = -25t^4\, dt$

$-\frac{du}{25} = t^4\, dt$

$\int \sqrt[3]{u} \cdot -\frac{du}{25}$

$= -\frac{1}{25} \int u^{1/3}$

$= -\frac{1}{25} \cdot \frac{u^{4/3}}{4/3} + C$

$= -\frac{1}{25} \cdot \frac{3}{4} u^{4/3} + C$

$= -\frac{3}{100} (3 - 5t^5)^{4/3}$

LESS APPARENT SUBSTITUTIONS

The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g'(x)$. If this is not the case, the method may still apply but may require more computation.

▶ **Example 9** Evaluate $\int x^2\sqrt{x-1}\,dx$.

$u = x - 1$

$du = dx$

$u = x - 1 \rightarrow u + 1 = x \rightarrow (u+1)^2 = x^2$

$(u^2 + 2u + 1) = x^2$

$\int (u^2 + 2u + 1)\sqrt{u}\,du$

$\int (u^2 + 2u + 1)\,u^{1/2}\,du$

$\int u^{2+1/2} + 2u^{1+1/2} + u^{1/2}\,du$

$\int u^{5/2}du + 2\int u^{3/2}du + \int u^{1/2}du$

$$\frac{u^{5/2+1}}{5/2+1} + 2\,\frac{u^{3/2+1}}{3/2+1} + \frac{u^{1/2+1}}{1/2+1} + C$$

$$\frac{u^{7/2}}{7/2} + 2\,\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C$$

$$\frac{2}{7}u^{7/2} + 2 \cdot \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$\frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

$$\lim_{x \to 2} \frac{\cos(\pi/x)}{x-2} \quad , \quad t = \frac{\pi}{2} - \frac{\pi}{x}$$

$$t = \frac{\pi}{2} - \frac{\pi}{x} \Rightarrow \frac{\pi}{x} = \frac{\pi}{2} - t$$

$$\frac{\pi}{x} = \frac{\pi}{2} - t$$

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t$$

$$\cos\left(\frac{\pi - 2t}{2}\right) = \sin t$$

$$\frac{\pi - 2t}{2}$$

$$\lim_{t \to 0} \frac{(\pi - 2t)\sin t}{4t}$$