Section (4.5) :

RIEMANN SUMS AND THE DEFINITE INTEGRAL

4.5.1 DEFINITION A function f is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.

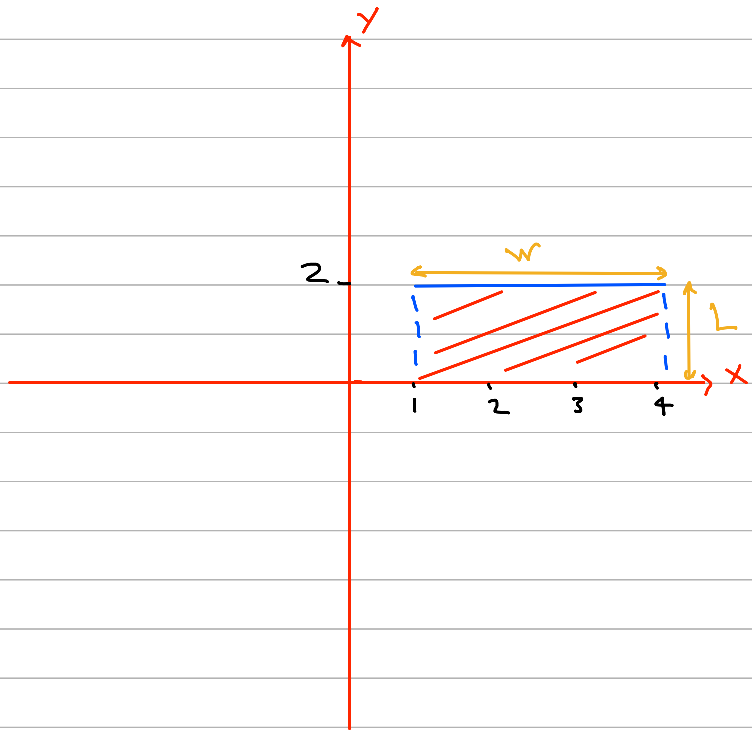
4.5.2 THEOREM *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)\,dx \tag{1}$$

▸ Example 1 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

(a) $\int_1^4 2\,dx$

$Y = 2$



$\int_1^4 2\,dx$ = area of rectangular

= Length . width

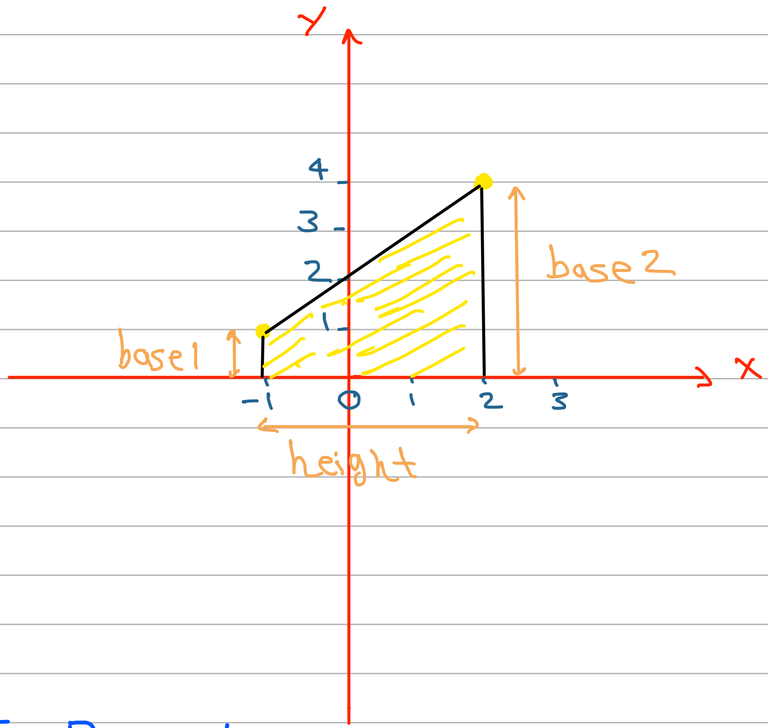
$= 2 . 3 = 6$

(b) $\int_{-1}^{2} (x+2)\,dx$

$Y = X + 2$

at $X = -1 \rightarrow Y = -1 + 2 = 1$ $(-1, 1)$

at $X = 2 \rightarrow Y = 2 + 2 = 4$ $(2, 4)$



$\int_{-1}^{2} (x+2)\,dx =$ area of Trapezoid

$= \frac{1}{2}(\text{Base1} + \text{Base2}) \cdot \text{height}$

$= \frac{1}{2}(1 + 4) \cdot 3$

$= \frac{1}{2}(5)(3) = \frac{15}{2}$

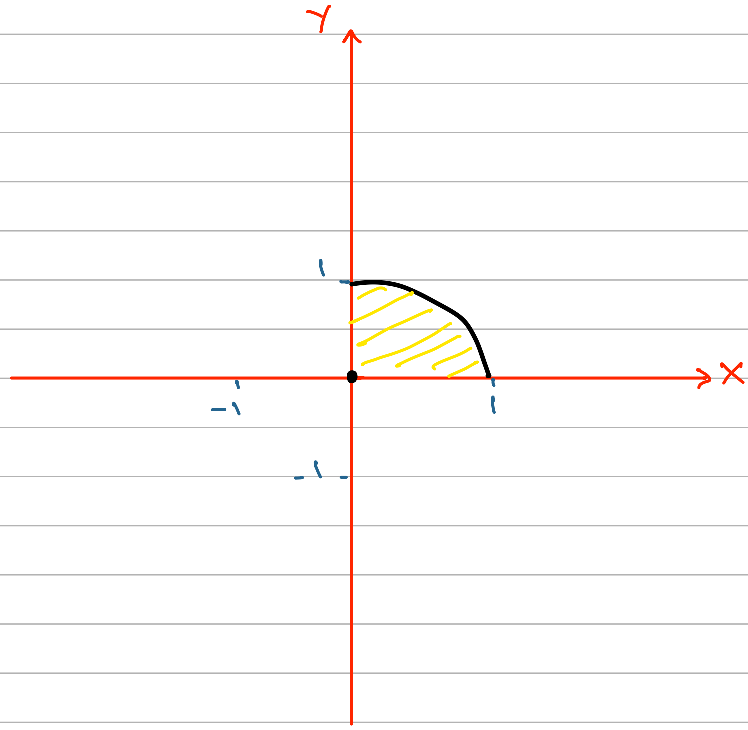
(c) $\int_0^1 \sqrt{1-x^2}\,dx$

$y = \sqrt{1-x^2}$

$(y)^2 = (\cancel{\sqrt{1-x^2}})^{\cancel{2}}$

$y^2 = 1 - x^2$

$y^2 + x^2 = 1$



$\int_0^1 \sqrt{1-x^2}\,dx =$ area of quarter circle

$= \frac{1}{4}\pi r^2$

$= \frac{1}{4}\pi(1)^2$

$= \frac{\pi}{4}$

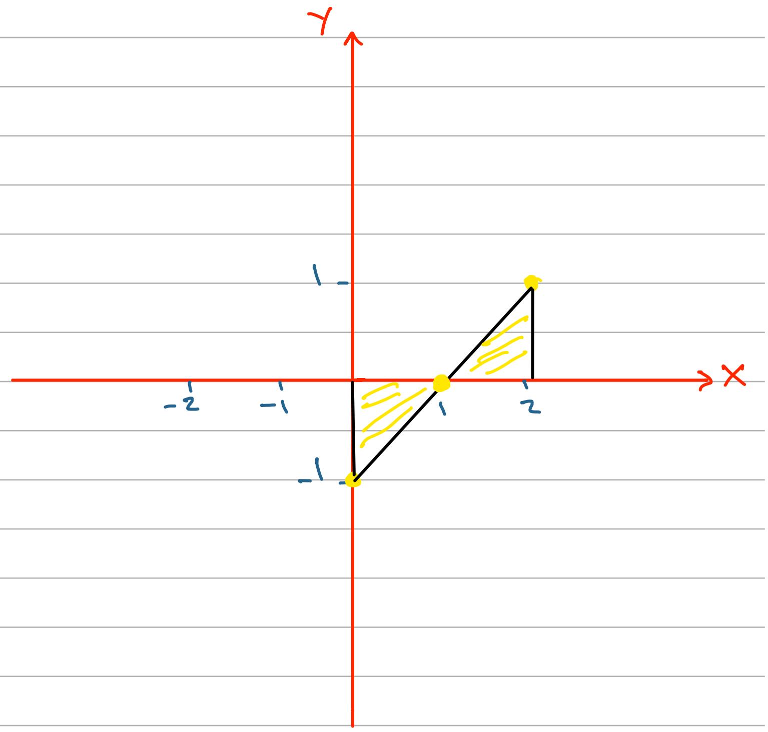
▶ **Example 2** Evaluate

(a) $\int_0^2 (x-1)\,dx$

$Y = X - 1$

at $X = 0 \longrightarrow Y = 0 - 1 = -1$ $(0,-1)$

at $X = 2 \longrightarrow Y = 2 - 1 = 1$ $(2,1)$



$\int_0^2 (x-1)\,dx = A_1 + (-A_2)$

$= A_1 - A_2$

$= \frac{1}{2}(\text{base1} \cdot \text{height1}) - \frac{1}{2}(\text{base2} \cdot \text{height2})$

$= \frac{1}{2}(1 \cdot 1) - \frac{1}{2}(1 \cdot 1)$

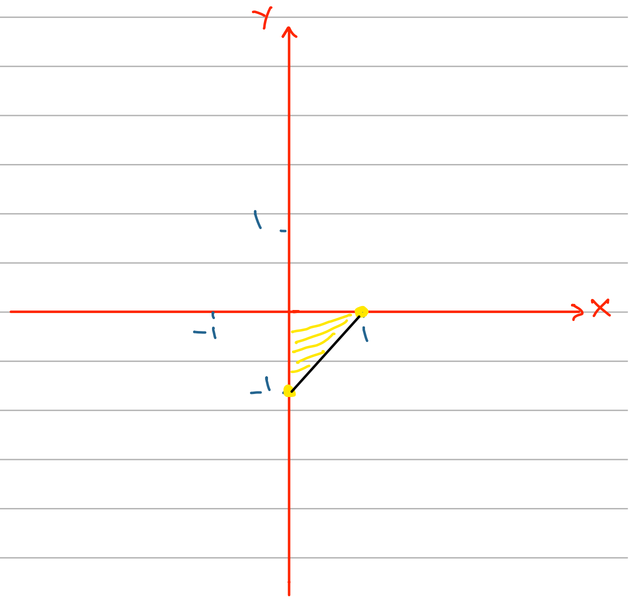
$= \frac{1}{2} - \frac{1}{2} = 0$

(b) $\int_0^1 (x-1)\,dx$

$Y = X - 1$

at $X = 0 \rightarrow Y = 0 - 1 = -1$ $(0, -1)$

at $X = 1 \rightarrow Y = 1 - 1 = 0$ $(1, 0)$



$\int_0^1 (x - 1)\,dx = \frac{1}{2}(\text{base} \cdot \text{height})$

$= \frac{1}{2}(1 \cdot 1)$

$= \frac{1}{2}$

PROPERTIES OF THE DEFINITE INTEGRAL

4.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

▶ Example 3

(a) $\int_1^1 x^2\,dx = \left(\frac{x^3}{3}\right)_1^1$

$$= \left(\frac{1^3}{3} - \frac{1^3}{3}\right)$$

$$= \frac{1}{3} - \frac{1}{3} = 0$$

دائماً اذا كانت حدود التكامل متساوية اذاً الناتج يساوي صفر

(b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}$

$$= -\frac{\pi}{4}$$

اذا كانت حدود التكامل الحد اللي فوق اصغر من الحد اللي تحت اذا نقلب حدود التكامل و نضع اشارة سالبة قبل التكامل

4.5.4 THEOREM *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f + g$, and $f - g$ are integrable on $[a, b]$ and*

(a) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(c) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

▶ **Example 4** Evaluate

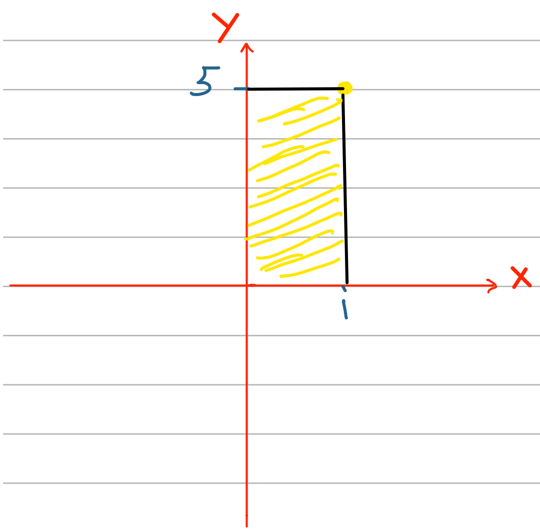
$$\int_0^1 (5 - 3\sqrt{1 - x^2})\,dx$$

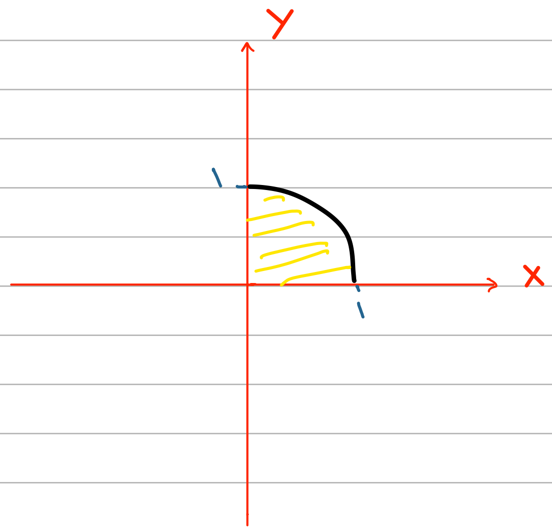
$$\int_0^1 5\,dx - 3\int_0^1 \sqrt{1 - x^2}\,dx$$

$$\int_0^1 5\,dx = (5x)\Big|_0^1 = (5(1) - 5(0)) = 5$$

$$\int_0^1 \sqrt{1 - x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

$$\int_0^1 5\,dx - 3\int_0^1 \sqrt{1 - x^2}\,dx = 5 - 3\left(\frac{\pi}{4}\right)$$





4.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

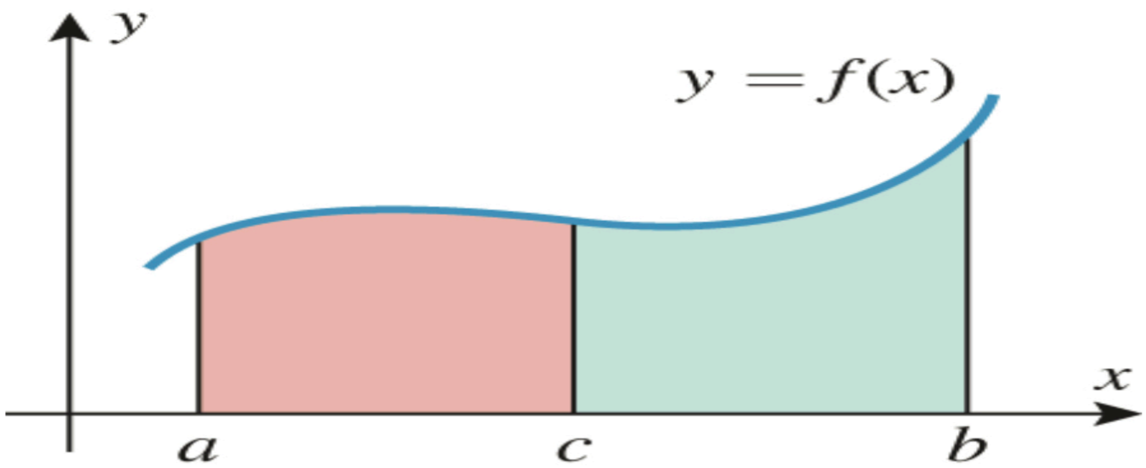


Figure 4.5.7

4.5.6 THEOREM

(*a*) *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq 0$$

(*b*) *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$